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Moving frames applied to shell elasticity

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Abstract
Exterior calculus and moving frames are used to describe the curved elasticity of shells. The kinematics follow from the Lie derivative on forms whereas the dynamics follow from stress forms.

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1. Introduction

Shell theory reduces the three-dimensional elasticity of a shell to an effective two-dimensional theory of its middle section. This paper presents a new short derivation of the shell equations of Kirchoff–Love based on the method of moving frames and exterior calculus.

Early shell theory dealt with church bells. Lord Rayleigh studied vibrating shells motivated by the question whether or not a full musical scale could be played on bells [1]. Vibrations of shells for aeroplanes and cars continue to be important in civil engineering. There, the numerical modelling of shells remain an active area of research. Also computer scientists have joined the efforts in order to present realistic computer graphics [2]. Lately, shell theory has been applied to cell membranes [3] and even nanotubes [4].

There are several approaches to shell theory [5, 6]. Kraus presents a variational method in [5]. Equivalently, assuming Newton's second law for an infinitesimal volume element and integrating over the thickness of the shell lead to the same results, see e.g. [6]. Unfortunately, the calculations are complicated: the dynamic part of this well-known law involves a covariant derivative of the stress tensor and requires intricate manipulations of Christoffel symbols for the full shell, respectively its middle section. More mathematical treatments of shell theory are found in [7–10].

However, a general observation is that complexities in tensor calculus can be avoided if it is possible to work in an orthonormal frame, i.e. in the case of Riemannian manifolds. This reduces the number of connection coefficients drastically and calculations become simpler and more transparent. For example, Maxwell's equations in electrodynamics and calculations of curvature in general relativity become much simpler when using the exterior calculus of differential forms and Cartan's method of moving frames [11]. The latter also have applications in soft-condensed matter to lipid membranes in [12, 13]. Likewise, the present paper shows how to apply exterior calculus to the continuum mechanics of shells.

1.1. Outline

The presentation centres around a proof of the shell equations of Kirchoff and Love [5]. These simple equations were chosen to illustrate the underlying technique. Some familiarity with differential forms and continuum mechanics is assumed. Section 2 introduces the geometry via differential forms, section 3 treats the kinematics of shell deformation and section 4 the equations of motion. A summary and some concluding remarks will be given in the last section. For convenience, supplementary material has been collected in appendices.

2. Geometry

2.1. Geometry of the mid-section $\mathcal{M}$

The shell equations must be covariant and therefore it is enough to validate them in one set of coordinates. Hence, on the mid-section $\mathcal{M}$ we can choose *lines of curvature* coordinates α^1 and α^2 having axes aligned with the principal directions of curvature. Such coordinates can always be chosen locally on a two-dimensional surface [14]. When the radii of curvature differ $R_1 \neq R_2$ the axes can be chosen uniquely. Otherwise, at an umbilic point $R_1 = R_2 \neq \infty$ or at a planar point $R_1 = R_2 = \infty$, we shall assume a choice made. Write in these coordinates the *mid-section metric* in the Lamé form

$$\mathrm{d}s^2 = A_1^2(\mathrm{d}\alpha^1)^2 + A_2^2(\mathrm{d}\alpha^2)^2. \tag{1}$$

A_i are called Lamé parameters. Instead of working directly with the coordinates α^1 and α^2 as done traditionally we shall work with a frame derived from these particular coordinates. Thus, define an orthonormal frame on the mid-section via the forms

$$\phi^a = A_a\,\mathrm{d}\alpha^a. \tag{2}$$

Then the metric of the mid-section, *the first fundamental tensor*, takes the simple form

$$\mathbf{a} = g_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b \equiv \phi^1 \otimes \phi^1 + \phi^2 \otimes \phi^2 \equiv \delta_{ab}\phi^a \otimes \phi^b \tag{3}$$

and the curvature tensor of the mid-section, *the second fundamental tensor*, becomes

$$\mathbf{d} = d_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b \equiv \frac{1}{R_1}\phi^1 \otimes \phi^1 + \frac{1}{R_2}\phi^2 \otimes \phi^2. \tag{4}$$

The corresponding dual tangent vectors are

$$\widetilde{\mathbf{e}_b} = \frac{1}{A_b}\frac{\partial}{\partial\alpha^b}. \tag{5}$$

2.2. Geometry of the shell $\mathcal{S}$

Denote the coordinates on the *three-dimensional shell* $\mathcal{S}$ as α^1, α^2 and z. The latter coordinate is perpendicular to the middle section of the shell, see figure 1. The direction of the z-axis follows that of [5] but is opposite from e.g., [6, 15]. The thickness of the shell is called h. $z = \pm h/2$ defines the upper respective the lower section of the shell. Likewise $z = 0$ corresponds to the mid-section $\mathcal{M}$. The *shell metric* is

$$\mathrm{d}s^2 = A_1^2(1 + z/R_1)^2(\mathrm{d}\alpha^1)^2 + A_2^2(1 + z/R_2)^2(\mathrm{d}\alpha^2)^2 + \mathrm{d}z^2. \tag{6}$$

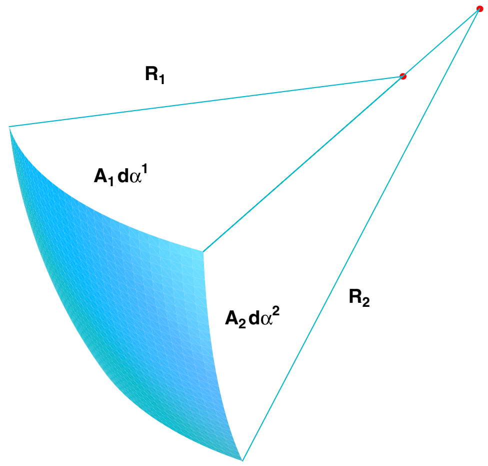

Figure 1. Middle section segment with curvature radii R_1, R_2 and increments $A_1\,\mathrm{d}\alpha^1$, $A_2\,\mathrm{d}\alpha^2$. (This figure is in colour only in the electronic version)

The factors $1 + z/R_i$ are understood as follows: an increment on the middle surface in the direction i spans the angle

$$\mathrm{d}\varphi^i = A_i\,\mathrm{d}\alpha^i/R_i$$

so the increment at the elevated position is

$$\mathrm{d}x^i(z) = (R_i + z)\,\mathrm{d}\varphi^i = A_i(1 + z/R_i)\,\mathrm{d}\alpha^i.$$

Finally, we shall assume the existence of derivatives of A_i, R_j.

2.2.1. Orthonormal frame of $\mathcal{S}$. We introduce a three-dimensional frame as done for the mid-section instead of working with the coordinates α^1, α^2 and z as in [5, 16] . Thus, from (6) define an orthonormal basis of 1-forms as

$$\boldsymbol{\Theta}^1 = A_1(1 + z/R_1)\,\mathrm{d}\alpha^1, \qquad \boldsymbol{\Theta}^2 = A_2(1 + z/R_2)\,\mathrm{d}\alpha^2 \qquad \text{and} \qquad \boldsymbol{\Theta}^3 = \mathrm{d}z. \tag{7}$$

From the 1-forms construct the volume element

$$\mathbf{dV} = \boldsymbol{\Theta}^1 \wedge \boldsymbol{\Theta}^2 \wedge \boldsymbol{\Theta}^3 = A_1 A_2 (1 + z/R_1)(1 + z/R_2)\,\mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2 \wedge \mathrm{d}z. \tag{8}$$

The Hodge dual forms to the 1-forms are oriented area elements

$$\mathbf{dS}_a = \star\boldsymbol{\Theta}^a \tag{9}$$

and fulfil

$$\boldsymbol{\Theta}^a \wedge \mathbf{dS}_b = \delta^a_b\,\mathbf{dV}. \tag{10}$$

In component form

$$\mathbf{dS}_i = \varepsilon_{ijk}\boldsymbol{\Theta}^j \otimes \boldsymbol{\Theta}^k \tag{11}$$

with ε_{ijk} being the totally antisymmetric tensor. Note, in the following we reserve ϵ_{ij} for another tensor, the strain, and hopefully no confusion will arise. As the area forms are not closed $\mathrm{d}(\mathbf{dS}_i) \neq 0$ they are not exact $\mathbf{dS}_i \neq \mathrm{d}$(1-form), and hence the notation of these area forms is slightly misleading although conventional.

The dual tangent vectors are likewise written as

$$\mathbf{e}_a = \frac{1}{(1 + z/R_a)A_a}\frac{\partial}{\partial\alpha^a}. \tag{12}$$

2.2.2. The connection of $\mathcal{S}$. The 1-forms of the frame lead to the connection coefficients via the condition of vanishing torsion [11]:

$$\mathrm{d}\Theta^i + \omega^i{}_j \wedge \Theta^j = 0. \tag{13}$$

Calculating $\mathrm{d}\Theta^i$ from (7), the connection form $\omega^i{}_j$ is found by inspection:

$$\omega = \left(\omega^i{}_j\right) = \begin{pmatrix} 0 & \frac{(A_1(1+z/R_1))_{,2}\Theta^1-(A_2(1+z/R_2))_{,1}\Theta^2}{A_1A_2(1+z/R_1)(1+z/R_2)} & \frac{\Theta^1}{R_1+z} \\ \frac{-(A_1(1+z/R_1))_{,2}\Theta^1+(A_2(1+z/R_2))_{,1}\Theta^2}{A_1A_2(1+z/R_1)(1+z/R_2)} & 0 & \frac{\Theta^2}{R_2+z} \\ -\frac{\Theta^1}{R_1+z} & -\frac{\Theta^2}{R_2+z} & 0 \end{pmatrix}, \tag{14}$$

where derivatives are denoted with a comma $\partial f/\partial\alpha^i = f_{,i}$. The Gauss–Codazzi equations discussed in appendix B allow us to simplify this result to

$$\omega = \begin{pmatrix} 0 & \frac{A_{1,2}\Theta^1/(1+z/R_1)-A_{2,1}\Theta^2/(1+z/R_2)}{A_1A_2} & \frac{\Theta^1}{R_1+z} \\ \frac{-A_{1,2}\Theta^1/(1+z/R_1)+A_{2,1}\Theta^2/(1+z/R_2)}{A_1A_2} & 0 & \frac{\Theta^2}{R_2+z} \\ -\frac{\Theta^1}{R_1+z} & -\frac{\Theta^2}{R_2+z} & 0 \end{pmatrix}. \tag{15}$$

2.2.3. Covariant derivative. Having defined the connection ω, the covariant derivative acting on forms is

$$\nabla = \mathbf{d} + \omega, \tag{16}$$

where $\mathbf{d}$ is the flat exterior derivative and ω acts from the left with the wedge product. From the condition of no torsion (13) follows that the covariant derivative vanishes on frame co-vectors respective tangent vectors. The latter holds since the metric is covariantly constant. Thus

$$\nabla\Theta^i = 0. \tag{17}$$

To exploit this structure we shall consequently define quantities of interest as forms.

2.2.4. The derived connection of $\mathcal{M}$. The mid-section $\mathcal{M}$ is embedded in the shell $\mathcal{S}$:

$$i : \mathcal{M} \to \mathcal{S}, \tag{18}$$

so restricting differential forms to the mid-section is the *pull-back*

$$i^* : \Lambda^*(\mathcal{S}) \to \Lambda^*(\mathcal{M}). \tag{19}$$

This restriction is obtained by putting $z = 0$. For instance,

$$i^*(\Theta^a) = \phi^a \qquad \text{and} \qquad i^*\Theta^3 = 0. \tag{20}$$

Likewise, pulling back the connection of the three-dimensional shell $\omega^i{}_j$ to the mid-section i^* gives *the* two-dimensional connection $\widetilde{\omega}^a_b$ of $\mathcal{M}$. Namely,

$$\mathrm{d}\phi^a = -\omega^a{}_b|_{z=0} \wedge \phi^b \equiv -\widetilde{\omega}^a_b \wedge \phi^b \tag{21}$$

with $a, b = 1, 2$ only, where the 'tilde' shall be used in the following when referring to the mid-section. Formally (21) holds since the exterior derivative commutes with the pull-back

$$di^* = i^*d \tag{22}$$

and

$$\mathrm{d}\phi^a = \mathrm{d}(i^*\Theta^a) = i^*\mathrm{d}\Theta^a = i^*\left(-\omega^a{}_i \wedge \Theta^i\right) = -i^*\left(\omega^a{}_b\right) \wedge \phi^b. \tag{23}$$

Thus, all connection coefficients of the mid-section $\widetilde{\Gamma}^i_{jk}$ are contained in the single 1-form

$$\widetilde{\omega}^1_2 = \frac{A_{1,2}\phi^1 - A_{2,1}\phi^2}{A_1 A_2}, \tag{24}$$

where

$$\widetilde{\omega}^a_b \equiv \widetilde{\Gamma}^a_{cb}\phi^c = -\widetilde{\omega}^b_a. \tag{25}$$

This corresponds to just two independent coefficients

$$\widetilde{\Gamma}^1_{12} = \frac{A_{1,2}}{A_1 A_2} \qquad \text{and} \qquad \widetilde{\Gamma}^1_{22} = -\frac{A_{2,1}}{A_1 A_2}. \tag{26}$$

Finally, these connection coefficients allow us to define the covariant derivative on the mid-section, $\widetilde{\nabla}$.

3. Shell kinematics

First we discuss the displacement field $\mathbf{U}$ and next the appropriate deformation field, the strain field $\boldsymbol{\epsilon}$.

3.1. The displacement field U

The displacement is a tangent vector field and hence contravariant. According to Kirchoff–Love (see appendix A):

$$\mathbf{U} = U^1\mathbf{e}_1 + U^2\mathbf{e}_2 + W\mathbf{e}_3 \equiv (u^1 + z\beta^1)\,\mathbf{e}_1 + (u^2 + z\beta^2)\,\mathbf{e}_2 + w\mathbf{e}_3 \tag{27}$$

with $\mathbf{e}_i$ given by (12). If $\boldsymbol{\beta}$ in (27) is sufficiently small its components play the role of angles. Thus, when seen from the middle section the displacement at height z in the direction i equals $z\,\mathrm{d}\varphi^i \approx z\beta^i$. However, in Kirchoff–Love's theory, the rotations β^i are not considered as independent fields but can be found from u^i and the gradient of w. This we show in section 3.3 using the assumption of vanishing normal strain $\epsilon_{i3} = 0$.

3.2. Strain field: connection to the Lie derivative

In linear elasticity the strain field $\boldsymbol{\epsilon}$ is often introduced as the linearized variation of the metric under a deformation [17]. This variation can be formulated strictly as the Lie derivative of the metric [18]. Eventually, it gives the strain as the symmetrized part of the deformation gradient. Thus,

$$\boldsymbol{\epsilon} = \tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{g}) = \tfrac{1}{2}(\boldsymbol{\nabla}\mathbf{U} + \mathbf{U}\boldsymbol{\nabla}). \tag{28}$$

We recall this fact in appendix C.

3.3. Lie derivative gives variation of curvature

We wish to emphasize the connection of the strain in shells to the variation of the second fundamental tensor. This is conveniently done via the Lie derivative instead of the symmetrized gradient in (28).

Expanding the shell metric. To find the strain with respect to the mid-section the full shell metric (6) is written out in terms of simpler mid-section 1-forms

$$\begin{aligned}\mathbf{g} &= \phi^1 \otimes \phi^1 + \phi^2 \otimes \phi^2 + \mathbf{\Theta}^3 \otimes \mathbf{\Theta}^3 + 2z\left(\frac{1}{R_1}\phi^1 \otimes \phi^1 + \frac{1}{R_2}\phi^2 \otimes \phi^2\right)\\ &\quad + z^2\left(\frac{1}{R_1^2}\phi^1 \otimes \phi^1 + \frac{1}{R_2^2}\phi^2 \otimes \phi^2\right)\\ &\equiv \mathbf{a} + \mathbf{\Theta}^3 \otimes \mathbf{\Theta}^3 + 2z\,\mathbf{d} + z^2\mathbf{d}^2 \end{aligned} \tag{29}$$

with $d^2_{ab} \equiv d_{ac}d^c_b$. Deriving with the Lie derivative gives

$$\begin{aligned}\epsilon &\equiv \tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{g})\\ &= \tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{a}) + \tfrac{1}{2}(\nabla w \otimes \mathbf{\Theta}^3 + \mathbf{\Theta}^3 \otimes \nabla w) + \mathcal{L}_{\mathbf{U}}(z\mathbf{d}) + \tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(z^2\mathbf{d}^2)\\ &= \tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{a}) + w\mathbf{d} + z\mathcal{L}_{\mathbf{U}}(\mathbf{d}) + zw\,\mathbf{d}^2 + \tfrac{1}{2}(\nabla w \otimes \mathbf{\Theta}^3 + \mathbf{\Theta}^3 \otimes \nabla w) + O(z^2). \end{aligned} \tag{30}$$

Above, the Lie derivative contracts the mid-section forms ϕ^a via the interior derivative $\iota_{\mathbf{U}}$ according to Cartan's formula for the Lie derivative on forms

$$\mathcal{L}_{\mathbf{U}}(\omega) = (\mathbf{d}\iota_{\mathbf{U}} + \iota_{\mathbf{U}}\,\mathbf{d})\omega \qquad \text{with} \qquad \omega \in \Lambda^*(\mathcal{S}). \tag{31}$$

Hence, it is advantageous to express the frame coordinates of the displacement field with respect to the mid-section:

$$\begin{aligned}(u^i + z\beta^i)e_i &\equiv (u^i + z\beta^i)\frac{1}{(1 + z/R_i)A_i}\frac{\partial}{\partial\alpha^i}\\ &= (u^i + z\beta^i)\big(\delta_i^j - zd_i^j\big)\frac{1}{A_j}\frac{\partial}{\partial\alpha^j} + O(z^2)\\ &\equiv \big(u^i + z\big(\beta^i - d^i_j u^j\big)\big)\tilde{e}_i + O(z^2) \equiv (\widetilde{u}^i + z\widetilde{\beta}^i)\tilde{e}_i + O(z^2) \end{aligned} \tag{32}$$

with

$$\widetilde{\mathbf{u}} = \mathbf{u} \tag{33}$$

$$\widetilde{\beta} = \beta - \mathbf{d}\cdot\mathbf{u}. \tag{34}$$

To proceed consider the first term in (30)

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{a}) = \tfrac{1}{2}\mathcal{L}_{\mathbf{u}+z\beta}(\mathbf{a}) = \tfrac{1}{2}\mathcal{L}_{\mathbf{u}}(\mathbf{a}) + \tfrac{1}{2}\mathcal{L}_{z\beta}(\mathbf{a}). \tag{35}$$

The first term in (35) reduces to a membrane strain expressed in mid-section coordinates

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{a}) = \tfrac{1}{2}(\widetilde{u}_{a;b} + \widetilde{u}_{b;a})\phi^a \otimes \phi^b = \tfrac{1}{2}(\widetilde{\nabla} \otimes \mathbf{u} + \mathbf{u} \otimes \widetilde{\nabla}) \tag{36}$$

with the corresponding covariant derivative. The second term in (35) gives a normal strain plus a strain similar to (36) of order $O(z)$:

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_{z\beta}(\mathbf{a}) &= \tfrac{1}{2}\widetilde{\beta}^a(\mathrm{d}z \otimes \phi^b + \phi^b \otimes \mathrm{d}z)\delta_{ab} + z\tfrac{1}{2}(\widetilde{\nabla} \otimes \widetilde{\beta} + \widetilde{\beta} \otimes \widetilde{\nabla})\\ &= \tfrac{1}{2}(\mathbf{\Theta}^3 \otimes \widetilde{\beta} + \widetilde{\beta} \otimes \mathbf{\Theta}^3) + z\tfrac{1}{2}(\widetilde{\nabla} \otimes \widetilde{\beta} + \widetilde{\beta} \otimes \widetilde{\nabla}). \end{aligned} \tag{37}$$

As $\frac{\partial w}{\partial z} = 0$ the normal strain ϵ_{33} vanishes in (30). So effectively

$$\nabla w = \mathrm{d}w = \mathrm{d}w(\alpha^1, \alpha^2) = \widetilde{\nabla} w. \tag{38}$$

Then the requirement of *vanishing normal shear strain* $\epsilon_{a3} = 0$ with (30) and (37) reduces to

$$\widetilde{\beta} = -\widetilde{\nabla} w + O(z). \tag{39}$$

An $O(z)$-correction is indicated, since $\mathcal{L}_{z\beta}(\mathbf{d})$ from $\mathcal{L}_{\mathbf{U}}(\mathbf{d})$ produces such a term. In conclusion

$$\begin{aligned}\epsilon \equiv \epsilon^{(0)} + z\epsilon^{(1)} &= \tfrac{1}{2}(\tilde{\nabla} \otimes \mathbf{u} + \mathbf{u} \otimes \tilde{\nabla}) + w\mathbf{d} \\ &\quad - z\left(\tfrac{1}{2}(\tilde{\nabla} \otimes \tilde{\nabla} w + \tilde{\nabla} w \otimes \tilde{\nabla}) - \mathbf{d}^2 w - \mathcal{L}_{\mathbf{u}}(\mathbf{d})\right) + O(z^2) \\ &= \tfrac{1}{2}(\tilde{\nabla} \otimes \mathbf{u} + \mathbf{u} \otimes \tilde{\nabla}) + w\mathbf{d} \\ &\quad - z(\tilde{\nabla} \otimes \tilde{\nabla} w - \mathbf{d}^2 w - \mathcal{L}_{\mathbf{u}}(\mathbf{d})) + O(z^2). \end{aligned} \tag{40}$$

This result agrees with that in [15] (where $\mathbf{d} := -\mathbf{d}$) when the Lie derivative in (40) is written in full. Using [19] or performing similar calculations as in appendix C gives the standard result

$$\mathcal{L}_{\mathbf{u}}(\mathbf{d}) = \left(d_{ab;c}u^c + d_a^c u_{b;c} + d_b^c u_{a;c}\right) \phi^a \otimes \phi^b \tag{41}$$

with the covariant derivative corresponding to the mid-section. This Lie derivative now measures the variation of the second fundamental tensor, the curvature tensor, with respect to membrane fields. It represents the change of curvature when only stretching.

First versus second fundamental tensor. To summarize, the Lie derivative of the first fundamental tensor a_{ab} produces parts of the leading in-plane strain $\epsilon^{(0)}$, whereas on the second fundamental tensor d_{ab} parts of the bending tensor $\epsilon^{(1)}$. Various measures of variation of the second fundamental tensor have been introduced and linearized in [6, 15]. The same results are obtained here formally, where the Lie derivative eventually acts on the curvature tensor in the expansion of the metric.

How unique is the bending tensor?. The discussion in this text considers the variation of $\mathbf{d} = d_{ab}\phi^a \otimes \phi^b$ under deformation. Since the Lie derivative does not commute with raising and lowering indices, in fact $\mathcal{L}_{\mathbf{U}}(\mathbf{g}) = 2\epsilon$, the various measurements of bending defined from the variation of d_{ab}, d_a^b or d^{ab} differ with terms of the form $\epsilon \cdot \mathbf{d}$, see [15]. For instance [6] considers the variation of d_b^c.

4. Shell dynamics

In this section, the stresses and moments in the shell are found. We follow [6] but use exterior calculus instead.

4.1. Force conservation in the bulk

In continuum mechanics, forces acting on general surface elements $\mathbf{dS}$ are described in terms of the stress tensor $\boldsymbol{\sigma}$ [17, 20]. Thus, the infinitesimal force on a surface element equals

$$\mathbf{df} = \mathbf{dS} \cdot \boldsymbol{\sigma}. \tag{42}$$

In the bulk, the force per unit volume becomes

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{U}) + \mathbf{X} = \rho \frac{\partial^2}{\partial t^2}\mathbf{U}, \tag{43}$$

where $\mathbf{X}$ is an external volume body force. In this section, we focus mainly on the left-hand side of (43), that is, the *dynamic* part of the equations of motion.

In exterior calculus, however, it is convenient to collect the forces on an infinitesimal element via the *stress form* [18]

$$\mathbf{f}^i = (\mathbf{dS} \cdot \boldsymbol{\sigma})^i = \sigma^{1i}\mathbf{dS}_1 + \sigma^{2i}\mathbf{dS}_2 + \sigma^{3i}\mathbf{dS}_3. \tag{44}$$

This form is a 2-form and when integrated over a surface it gives the corresponding force (here in direction i, which by convention is put as the last index in σ^{ai}). The total force is written as

$$\mathbf{f} = \mathbf{f}^i \mathbf{e}_i \equiv \mathbf{f}^i \otimes \mathbf{e}_i. \tag{45}$$

Its covariant derivative is simple

$$\nabla \mathbf{f} = \nabla \mathbf{f}^i \otimes \mathbf{e}_i + \mathbf{f}^i \otimes \nabla \mathbf{e}_i = \nabla \mathbf{f}^i \otimes \mathbf{e}_i \tag{46}$$

as the frame and its dual are constant and similar for $\mathbf{X}$, $\ddot{\mathbf{u}}$. In differential form (43) reads for the component i

$$\nabla \mathbf{f}^i + \mathbf{X}^i \, \mathbf{dV} = \rho \ddot{u}^i \mathbf{dV}. \tag{47}$$

4.2. Covariant derivative of stress form

First we discuss tangential directions. Hence, make a split in tangential and normal (indices $a, b \in \{1, 2\}$ versus 3) in (47):

$$\begin{aligned}\nabla \mathbf{f}^a &= \nabla(\sigma^{1a}\mathbf{dS}_1 + \sigma^{2a}\mathbf{dS}_2 + \sigma^{3a}\mathbf{dS}_3)\\ &= \partial_1(\sigma^{1a}A_2(1 + z/R_2))\, \mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2 \wedge \mathrm{d}z + \partial_2(\sigma^{2a}A_1(1 + z/R_1))\, \mathrm{d}\alpha^2 \wedge \mathrm{d}z \wedge \mathrm{d}\alpha^1\\ &\quad + \partial_3(\sigma^{3a}(1 + z/R_1)(1 + z/R_2)A_1A_2)\, \mathrm{d}z \wedge \mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2 + \boldsymbol{\omega}^a{}_b \wedge \mathbf{f}^b + \boldsymbol{\omega}^a{}_3 \wedge \mathbf{f}^3.\end{aligned} \tag{48}$$

Consider the first component $a = 1$. A simplification occurs in the last two terms of (48), since e.g. $\boldsymbol{\omega}^a{}_3 = \boldsymbol{\Theta}^a/(R_a + z)$ restricts the relevant term in $\mathbf{f}^3$ to $\sigma^{a3}\mathbf{dS}_a$ by (10). Likewise the second last term has $b = 2$ by the antisymmetry of $\boldsymbol{\omega}$, where

$$\boldsymbol{\omega}^1{}_2 = \frac{1}{A_1A_2}\left(A_{1,2}\boldsymbol{\Theta}^1/(1 + z/R_1) - A_{2,1}\boldsymbol{\Theta}^2/(1 + z/R_2)\right) \tag{49}$$

picks out corresponding terms of $\mathbf{f}^2$, that is σ^{21} for the first term in (49) and σ^{22} for the second.

4.3. Momentum conservation

The next step is to eliminate the degree of freedom in the direction perpendicular to the shell by integrating (47) over the thickness coordinate, $\int_{z=-h/2}^{h/2}$. From (48), a typical term has the form of an integral of the stress tensor weighted with the curvature factors $1+z/R_i$. This leads to the concept of stress resultants which we shall briefly pause to define.

4.3.1. Stress resultants. Consider the force per length with respect to an edge along the middle section. For example, assume the edge is perpendicular to the first direction. That force density equals

$$N^{1a} \equiv \frac{\int_{-h/2}^{h/2} \mathrm{d}S_1(z)\sigma^{1a}}{A_2\, \mathrm{d}\alpha^2} = \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{1a}(1 + z/R_2). \tag{50}$$

The notation is such that the first index denotes the direction to which the perpendicular edge is considered. Thus, N^{1a} is the force density along the edge 2. Likewise $Q^2 \equiv N^{23}$ is the force density perpendicular to the shell in the z-direction along edge 1.

Thus in general the force line densities are

$$\begin{aligned}
N^{a} &= N^{aa} \\
N^{1a} &= \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{1a}(1 + z/R_2) \\
N^{2a} &= \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{2a}(1 + z/R_1) \\
Q^{1} &= \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{13}(1 + z/R_2) \\
Q^{2} &= \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{23}(1 + z/R_1)
\end{aligned} \tag{51}$$

referred to as *stress resultants*. As mentioned, these arise in the thickness integration of (47). Also corresponding moments are defined as

$$\begin{aligned}
M^{a} &= M^{aa} \\
M^{1a} &= \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{1a} z(1 + z/R_2) \\
M^{2a} &= \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{2a} z(1 + z/R_1)
\end{aligned} \tag{52}$$

following the conventions of [5].

The resultants satisfy the constraint

$$N^{12} - N^{21} = \frac{1}{R^2} M^{21} - \frac{1}{R_1} M^{12}. \tag{53}$$

Equation (53) follows from the symmetry of the stress tensor

$$0 = \int \mathrm{d}z(\sigma^{12} - \sigma^{21})(1 + z/R_1)(1 + z/R_2) = N^{12} + \frac{1}{R_1} M^{12} - N^{21} - \frac{1}{R_2} M^{21}. \tag{54}$$

We express the resultants as tensors on $\mathcal{M}$ as

$$\begin{aligned}
N^{ab} &= \int_{-h/2}^{h/2} \mathrm{d}z \big(\delta^a_c + z\check{d}^a_c\big)\sigma^{cb} \\
Q^{a} &= \int_{-h/2}^{h/2} \mathrm{d}z \big(\delta^a_c + z\check{d}^a_c\big)\sigma^{c3} \\
M^{ab} &= \int_{-h/2}^{h/2} \mathrm{d}z\, z \big(\delta^a_c + z\check{d}^a_c\big)\sigma^{cb}
\end{aligned} \tag{55}$$

where a modified curvature tensor $\check{\mathbf{d}}$ is introduced as

$$\check{\mathbf{d}} = \frac{1}{R_2}\phi^1 \otimes \phi^1 + \frac{1}{R_1}\phi^2 \otimes \phi^2 = (\mathrm{tr}\,\mathbf{d})\mathbf{a} - \mathbf{d}. \tag{56}$$

In (55) the tensor σ^{ab} transforms as well as tensor on $\mathcal{M}$ under transformations of $\mathcal{S}$ leaving z fixed. For other resultants, see [6]. The particular form of the resultants in terms of strains is discussed in appendix D.

4.3.2. Force balance for shell element. After having defined stress resultants, we continue the example from section 4.2. When integrating over z and multiplying with A_1A_2 note that A_a and R_a are independent of z. Also the symmetry $\sigma_{21} = \sigma_{12}$ is used to get a term N^{12}. The integration over the $\partial_3(= \partial_z)$-term is evaluated at the boundary. Collecting terms proportional to $\mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2$ gives the first equation of motion:

$$\partial_1(A_2N^1) + \partial_2(A_1N^{21}) + A_{1,2}N^{12} - A_{2,1}N^2 + A_1A_2(Q^1/R_1 + q_1) = A_1A_2\rho h\ddot{u}^1 \tag{57}$$

with the effective load

$$q^1 = [\sigma^{13}(1 + z/R_1)(1 + z/R_2)]_{z=-h/2}^{h/2} + \int_{-h/2}^{h/2} \mathrm{d}z\, X^1(1 + z/R_1)(1 + z/R_2). \tag{58}$$

Similar calculations are done for the second component $a = 2$ in (48).

4.4. Moment conservation

The moment in the direction $\mathbf{z} \wedge \mathbf{e}_i$ is found by multiplying (47) with z. As before we consider $i = a = 1$ and the derivation is very similar to the above and differs only in a partial integration of the corresponding ∂_3-term

$$\begin{aligned}&\int_{-h/2}^{h/2} \mathrm{d}z z\partial_3(\sigma^{31}(1 + z/R_1)(1 + z/R_2))A_1A_2\mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2\\ &\quad= \left([z\sigma^{31}(1 + z/R_1)(1 + z/R_2)]_{z=-h/2}^{h/2} - \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{31}(1 + z/R_2)\right)A_1A_2\mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2\\ &\quad- \int_{-h/2}^{h/2} \mathrm{d}z\, \sigma^{31}(1 + z/R_2)z/R_1A_1A_2\, \mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2\end{aligned} \tag{59}$$

The last term cancels with

$$\int_{z=-h/2}^{h/2} z\boldsymbol{\omega}^1{}_3 \wedge \mathbf{f}^3 = \int_{z=-h/2}^{h/2} z\boldsymbol{\omega}^1{}_3 \wedge \sigma^{31}\boldsymbol{\Theta}^2 \wedge \mathrm{d}z.$$

The second term in the parenthesis in (59) gives the integral for Q^1, whereas the boundary term is grouped together with body moments as done above for the effective load q^i.

4.5. Normal resultants

The final stress resultant involves the normal stress and is denoted by Q^a. The calculation can be done as in (57) and stated with derivatives of products of Lamé coefficients A_i and resultants. These derivatives can be expanded and simplified using covariant derivatives of the mid-section with the connection from (24).

However, this can also be seen at an earlier stage in the calculation by keeping the mid-section frame $\{\phi^1, \phi^2, \mathrm{d}z\}$ and evaluating derivatives using (21). For simplicity, we show this alternative approach for Q^a but similar calculations can be done for the other resultants. The covariant derivative becomes

$$\nabla\mathbf{f}^3 = \mathrm{d}\mathbf{f}^3 + \boldsymbol{\omega}^3{}_a \wedge \mathbf{f}^a, \tag{60}$$

where

$$\begin{aligned}\mathrm{d}\mathbf{f}^3 &= \mathrm{d}(\sigma^{13}\boldsymbol{\Theta}^2 \wedge \mathrm{d}z) + \mathrm{d}(\sigma^{23}\mathrm{d}z \wedge \boldsymbol{\Theta}^1) + \mathrm{d}(\sigma^{33}\boldsymbol{\Theta}^1 \wedge \boldsymbol{\Theta}^2)\\ &= (\partial_{\hat{1}}(\sigma^{13}(1 + z/R_2)) + \partial_{\hat{2}}(\sigma^{23}(1 + z/R_1)) + \partial_3(\sigma^{33}(1 + z/R_1)(1 + z/R_2)))\phi^1 \wedge \phi^2 \wedge \mathrm{d}z\\ &\quad- (\sigma^{13}(1 + z/R_2))\widetilde{\omega}_1^2 \wedge \phi^1 \wedge \mathrm{d}z + (\sigma^{23}(1 + z/R_1))\, \mathrm{d}z \wedge \widetilde{\omega}_2^1 \wedge \phi^2\\ &= \big[\partial_{\hat{1}}(\sigma^{13}(1 + z/R_2)) + \partial_{\hat{2}}(\sigma^{23}(1 + z/R_1)) + \partial_3(\sigma^{33}(1 + z/R_1)(1 + z/R_2))\\ &\quad+ (\sigma^{13}(1 + z/R_2))\widetilde{\Gamma}_{21}^2 + (\sigma^{23}(1 + z/R_1))\widetilde{\Gamma}_{12}^1\big]\, \phi^1 \wedge \phi^2 \wedge \mathrm{d}z\end{aligned} \tag{61}$$

with the derivations $\partial_{\hat{b}}$ corresponding to those of the mid-section (5) and

$$\boldsymbol{\omega}^3{}_a \wedge \mathbf{f}^a = -\frac{\boldsymbol{\Theta}^a}{R_a + z} \wedge \sigma^{ba}\,\mathbf{dS}_b = -\frac{1}{R_a + z}\sigma^{ba}\delta_{ab}\,\mathbf{dV}. \tag{62}$$

Hence

$$\begin{aligned}\int \nabla \mathbf{f}^3 &= \big(\partial_{\hat{1}} Q^1 + \widetilde{\Gamma}^1_{12} Q^2 + \partial_{\hat{2}} Q^2 + \widetilde{\Gamma}^2_{21} Q^1 - d_{ab} N^{ab} \\ &\quad + [\sigma^{33}(1 + z/R_1)(1 + z/R_2)]^{h/2}_{z=-h/2}\big)\phi^1 \wedge \phi^2 \\ &\equiv [\widetilde{\nabla}_a Q^a - d_{ab} N^{ab} + q^3]\phi^1 \wedge \phi^2,\end{aligned} \tag{63}$$

as $\widetilde{\Gamma}^i_{ji} \equiv \iota_{\widetilde{\mathbf{e}}_j}\widetilde{\boldsymbol{\omega}}^i_i = 0$ for all i, j (no summation) by the antisymmetry of the frame connection coefficients. On the other hand, the kinematic body force equals

$$\int_{z=-h/2}^{h/2} \rho \ddot{u}^3 \mathbf{dV} \equiv \rho h \ddot{w}\, \phi^1 \wedge \phi^2 + O(h^3). \tag{64}$$

Thus to leading order

$$\widetilde{\nabla}_a Q^a - d_{ab} N^{ab} + q^3 = \rho h \ddot{w}. \tag{65}$$

4.6. Inertia terms

We comment on the kinematic part of (43): when the acceleration displacement field $\ddot{\mathbf{u}}$ is integrated over thickness only those terms constant in z survive leading to terms of the form $\rho h \ddot{u}^a$ as in (57) and (65). But for the moment equations, (47) is already multiplied with z and only the acceleration of β^i remains. This leads to terms

$$\frac{h^3}{12} A_1 A_2 \rho \ddot{\beta}^i,$$

which go as the cube of thickness and are neglected in this treatment.

4.7. Dynamical equations of motion

In summary, using exterior calculus and moving frames one directly finds the shell equations in classical form [5]:

$$\begin{aligned}&\partial_1(A_2 N^1) + \partial_2(A_1 N^{21}) + A_{1,2} N^{12} - A_{2,1} N^2 + A_1 A_2 (Q^1/R_1 + q^1) = A_1 A_2 \rho h \ddot{u}^1 \\ &\partial_1(A_2 N^{12}) + \partial_2(A_1 N^2) + A_{2,1} N^{21} - A_{1,2} N^1 + A_1 A_2 (Q^2/R_2 + q^2) = A_1 A_2 \rho h \ddot{u}^2 \\ &\partial_1(A_2 Q^1) + \partial_2(A_1 Q^2) - A_1 A_2 (N^1/R_1 + N^2/R_2) + A_1 A_2 q^3 = A_1 A_2 \rho h \ddot{w} \\ &\partial_1(A_2 M^1) + \partial_2(A_1 M^{21}) + A_{1,2} M^{12} - A_{2,1} M^2 - A_1 A_2 (Q^1 - m^1) = 0 \\ &\partial_1(A_2 M^{12}) + \partial_2(A_1 M^2) + A_{2,1} M^{21} - A_{1,2} M^1 - A_1 A_2 (Q^2 - m^2) = 0.\end{aligned} \tag{66}$$

However, having calculated the connection coefficients in (24) (26), the equations for the resultants are easily rewritten in *covariant* form:

$$\begin{aligned}&\widetilde{\nabla}_b N^{ba} + d^a_b Q^b + q^a = \rho h \ddot{u}^a \\ &\widetilde{\nabla}_a Q^a - N^{ab} d_{ba} + q^3 = \rho h \ddot{w} \\ &\widetilde{\nabla}_b M^{ba} - Q^a + m^a = 0,\end{aligned} \tag{67}$$

since in (66) in the equations for N^{ab} and M^{ab} all the first four terms come from a tangential covariant derivative. Likewise for Q^a as demonstrated explicitly in section 4.5.

Eliminating the normal stress Q^a and assuming no body forces and moments gives

$$\begin{aligned} \widetilde{\nabla}_b N^{ba} + d^a_b \widetilde{\nabla}_c M^{cb} &= \rho h \ddot{u}^a \\ \widetilde{\nabla}_a \widetilde{\nabla}_b M^{ba} - N^{ab} d_{ba} &= \rho h \ddot{w}. \end{aligned} \tag{68}$$

As these equations hold in lines of curvature coordinates and are covariant they are true in any other coordinate system on the mid-section.

5. Summary and conclusion

The method of moving frames and exterior calculus allow a fast derivation of the equations of motion for a curved elastic shell. First, the kinematic stretching and bending field correspond to a particular strain tensor field obtained using the Lie derivative on the metric. Second, the equations describing the dynamics are found using stress forms which are differential 2-forms encoding the stress tensor in a convenient way. Finally, all equations are seen to be covariant and hence valid in any coordinate system.

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Appendix A. Thin shell assumptions

The assumptions of Kirchoff and Love for a thin shell with vertical coordinate z and intrinsic coordinates α_1, α_2 are as follows:

(i) thickness small compared to radii of curvature,
(ii) small displacements,
(iii) vanishing normal stress $\sigma_{33} = 0$,
(iv) preservation of normals $\epsilon_{i3} = 0$,
(v) linear dependence of membrane field $U^i = u^i(\alpha^1, \alpha^2, t) + z\beta^i(\alpha^1, \alpha^2, t)$,
(vi) constant dependence of flexural field $W = w(\alpha^1, \alpha^2, t)$.

Appendix B. Gauss–Codazzi

There exist constraints on the functions A_i and R_i in the metric (6) for defining a valid surface. These equations come about by expressing the curvature via the connection. Even though the shell is curved the metric (6) is just a metric for the flat three-dimensional space $\mathbb{R}^3$. Consequently, the curvature is zero:

$$0 = \boldsymbol{\Omega}^i{}_j = \mathrm{d}\boldsymbol{\omega}^i{}_j + \boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j. \tag{B.1}$$

For instance for $i = 1$ and $j = 3$

$$\mathrm{d}\boldsymbol{\omega}^1{}_3 = (A_1/R_1)_{,2}\, \mathrm{d}\alpha^2 \wedge \mathrm{d}\alpha^1 \tag{B.2}$$

and

$$\boldsymbol{\omega}^1{}_2 \wedge \boldsymbol{\omega}^2{}_3 = \frac{(A_1(1 + z/R_1))_{,2}}{R_2 + z} \mathrm{d}\alpha^1 \wedge \mathrm{d}\alpha^2, \tag{B.3}$$

so

$$\frac{(A_1(1 + z/R_1))_{,2}}{R_2} = \left(\frac{A_1}{R_1}\right)_{,2} (1 + z/R_2). \tag{B.4}$$

At the middle surface $z = 0$ the classical Codazzi equation is found

$$\frac{A_{1,2}}{R_2} = \left(\frac{A_1}{R_1}\right)_{,2}. \tag{B.5}$$

Applying (B.5) to (B.4) gives

$$(A_1(1 + z/R_1))_{,2} = A_{1,2}(1 + z/R_2) \tag{B.6}$$

used in (15). Likewise other identities between A_1, A_2, R_1 and R_2 hold generalizing the Gauss–Codazzi equations to the shell metric. By considering $i = 1$ and $j = 2$ the result of Gauss follows:

$$\left(\frac{A_{1,2}}{A_2}\right)_{,2} + \left(\frac{A_{2,1}}{A_1}\right)_{,1} = -\frac{A_1 A_2}{R_1 R_2}. \tag{B.7}$$

Hence using (B.7), the total curvature $K \equiv \frac{1}{R_1 R_2}$ related to the embedding may be expressed entirely in terms of the intrinsic functions A_i belonging to metric of the surface. This is the lines of curvature coordinates version of Gauss's theorema egregium [14].

Appendix C. Strain as Lie derivative

In linear elasticity the strain field is found from the displacement field by an application of a Lie derivative on the metric [18]:

$$\epsilon = \epsilon_{ij}\boldsymbol{\Theta}^i \otimes \boldsymbol{\Theta}^j = \tfrac{1}{2}\mathcal{L}_{\mathbf{U}}(\mathbf{g}) \qquad \text{with} \qquad \mathbf{g} \equiv g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = \delta_{jk}\boldsymbol{\Theta}^j \otimes \boldsymbol{\Theta}^k. \tag{C.1}$$

The Lie derivative $\mathcal{L}$ is with respect to the physical displacement field $\mathbf{U}$.

The strain field is calculated with respect to the orthonormal frame (7). As $\mathcal{L}$ is a derivation:

$$\mathcal{L}(\boldsymbol{\Theta}^j \otimes \boldsymbol{\Theta}^k) = \mathcal{L}(\boldsymbol{\Theta}^j) \otimes \boldsymbol{\Theta}^k + \boldsymbol{\Theta}^j \otimes \mathcal{L}(\boldsymbol{\Theta}^k), \tag{C.2}$$

it suffices to investigate its action on a co-vector $\boldsymbol{\Theta}^j$. Using Cartan's formula (31) for the Lie derivative and (17):

$$\begin{aligned}\mathcal{L}_{\mathbf{U}}(\boldsymbol{\Theta}^j) &\equiv (d\iota_{\mathbf{U}} + \iota_{\mathbf{U}} d)\boldsymbol{\Theta}^j = \mathrm{d}U^j + \iota_{\mathbf{U}}(d\boldsymbol{\Theta}^j) = \mathrm{d}U^j - \iota_{\mathbf{U}}\big(\omega^j{}_k \wedge \boldsymbol{\Theta}^k\big)\\ &= \mathrm{d}U^j + U^k\omega^j{}_k - \big(\iota_{\mathbf{U}}\omega^j{}_k\big)\boldsymbol{\Theta}^k = \nabla U^j - \big(\iota_{\mathbf{U}}\omega^j{}_k\big)\boldsymbol{\Theta}^k.\end{aligned} \tag{C.3}$$

The antisymmetry $\omega^j{}_k = -\omega^k{}_j$ gives the familiar result from continuum mechanics:

$$\epsilon = \tfrac{1}{2}(\nabla U^j \otimes \boldsymbol{\Theta}^k + \boldsymbol{\Theta}^k \otimes \nabla U^j)g_{jk} = \tfrac{1}{2}(\nabla \otimes \boldsymbol{\Theta}^j U_j + U_j\boldsymbol{\Theta}^j \otimes \nabla) = \tfrac{1}{2}(\nabla\mathbf{U} + \mathbf{U}\nabla). \tag{C.4}$$

In (C.4) the metric lowers the indices of the displacement field coordinates from a tangent vector to a co-vector. Sometimes this is made explicit by writing $\mathbf{U} = \mathbf{U}^\flat$.

Appendix D. Constitutive equations

Although all that we wanted to consider concerning moving frames and shells was presented in the previous sections we mention for completeness one final subject, which is how to relate the resultants with the displacements. Here a link between stress and strain is needed. These are in the form of *constitutive equations* and can be thought of as a generalization of Hooke's law.

D.1. Plane stress

In the analysis of plates and shell the so-called plane stress approximation is used [5, 6]. In that approximation, the stress tensor has almost the same form as in bulk elasticity except that the coefficients of elasticity are slightly altered. We refer the reader to [5, 6] for a full discussion and just state the approximation in the case of isotropic materials

$$\sigma^{ab} = \frac{E\nu}{1-\nu^2} a^{ab} \operatorname{tr}\epsilon + \frac{E}{1+\nu}\epsilon^{ab}. \tag{D.1}$$

Following the notation of [21] a 4-tensor is introduced

$$H^{abcd} = \frac{1-\nu}{2}\left(a^{ac}a^{bd} + a^{ad}a^{bc}\right) + \nu a^{ab}a^{cd} \tag{D.2}$$

such that

$$\sigma^{ab} = \frac{E}{1-\nu^2} H^{abcd}\epsilon_{cd}. \tag{D.3}$$

D.2. Stress resultants

We are now prepared to integrate (51) to get the resultants. Separate the stress tensor according to the different orders in the expansion in z:

$$\sigma^{ab(i)} = \frac{E}{1-\nu^2} H^{abcd}\epsilon^{(i)}_{cd} \tag{D.4}$$

for $i = 0, 1$ with $\epsilon^{(i)}$ given by (40). Plain calculation from (55) and (D.1) gives the resultants as

$$\begin{aligned} N^{ab} &= h\sigma^{ab(0)} + \frac{h^3}{12}\check{d}^a_c\sigma^{cb(1)} \\ M^{ab} &= \frac{h^3}{12}\left(\sigma^{ab(1)} + \check{d}^a_c\sigma^{cb(0)}\right) \end{aligned} \tag{D.5}$$

using $\check{\mathbf{d}}$ defined by (56). Or in terms of the plane stress tensor H^{abcd}

$$\begin{aligned} N^{ab} &= CH^{abcf}\epsilon^{(0)}_{cf} + B\check{d}^a_c H^{cbfe}\epsilon^{(1)}_{fe} \\ M^{ab} &= B\left(H^{abcf}\epsilon^{(1)}_{cf} + \check{d}^a_c H^{cbfe}\epsilon^{(0)}_{fe}\right). \end{aligned} \tag{D.6}$$

The constants C and B are the stretching and bending rigidities

$$C = \frac{Eh}{1-\nu^2} \qquad \text{with} \qquad B = \frac{Eh^3}{12(1-\nu^2)}. \tag{D.7}$$

In (D.5) the factors z/R_i from (55) are included leading to corrections proportional to h^3/R_i, see [5, 6]. If included relation (53) among the resultants is satisfied. Thus, some tensor algebra on (D.5) shows

$$\varepsilon_{ab}N^{ab} = -\varepsilon_{ab}d^a_c M^{cb}. \tag{D.8}$$

Therefore, the antisymmetric part of $\mathbf{N}$ equals that of $\mathbf{d}\cdot\mathbf{M}$ and in lines of curvature coordinates (D.8) has the form

$$N^{12} - N^{21} = \frac{1}{R_2}M^{21} - \frac{1}{R_1}M^{12} \tag{D.9}$$

which coincides with (53).

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